One refinement of Weitzenböck's inequality

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Prove that in any $\triangle ABC$

$$a^2+b^2+c^2 \geq 4S\sqrt{\frac{1}{2}(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-\frac{3}{2}}$$

where S is area of $\triangle ABC$.

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We will denote area of $\triangle ABC$ via F and let R, r, s be circumradius. inradius and semiperimeter of $\triangle ABC$. Let $x := s-a, y := s-b, z := s-c, p := xy+yz+zx, q := xyz$. Then, assuming $s = 1$ (due homogeneity) we obtain $x, y, z > 0, x+y+z = 1, a = 1-x$, $b = 1-y, c = 1-z$, $r = \sqrt{\frac{1}{s}\prod(x-a)} = \sqrt{q}, abc = p-q$, $F = \sqrt{q}$,

$ab+bc+ca = 1+p, a^2+b^2+c^2 = 2(1-p)$ and $a^2b^2+b^2c^2+c^2a^2 =$
$(ab+bc+ca)^2-2abc(a+b+c) = (1+p)^2-4(p-q) = (1-p)^2+4q.$

and, therefore,

$$(a^2+b^2+c^2)^2-16F^2\cdot\left(\frac{(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)}{2a^2b^2c^2}-\frac{3}{2}\right) =$$

$$4(1-p)^2-16\cdot q\left(\frac{2(1-p)\left((1-p)^2+4q\right)}{2(p-q)^2}-\frac{3}{2}\right).$$

Since $3p = 3(xy+yz+zx) \leq (x+y+z)^2 = 1$ and $p > 0$ then $t := \sqrt{1-3p}, p = \frac{1-t^2}{3}$ and $p \in (0, 1/3] \Leftrightarrow t \in [0,1)$. Also note that $q = xyz \leq \frac{x+y+z}{3}\cdot\frac{xy+yz+zx}{3} = \frac{p}{9}$.

Since $q\left(\frac{2(1-p)\left((1-p)^2+4q\right)}{2(p-q)^2}-\frac{3}{2}\right)$ increase by $q \in (0, p/9]$

and * $q \leq \frac{(1-t)^2(1+2t)}{27}$ (it is the best upper bound for q) then

$$4(1-p)^2-16\cdot q\left(\frac{2(1-p)\left((1-p)^2+4q\right)}{2(p-q)^2}-\frac{3}{2}\right) \geq 4\left(\frac{2+t^2}{3}\right)^2-$$

$$16\cdot\frac{(1-t)^2(1+2t)}{27}\left(\frac{2\cdot\left(\frac{2+t^2}{3}\right)\left(\left(\frac{2+t^2}{3}\right)^2+4\cdot\frac{(1-t)^2(1+2t)}{27}\right)}{2\left(\frac{1-t^2}{3}-\frac{(1-t)^2(1+2t)}{27}\right)^2}-\frac{3}{2}\right) =$$

$$\frac{4}{9}t^2(t-1)^2\frac{(t-4)^2}{(t+2)^2} \geq 0.$$

* Let $t \in [0,1)$ and $p = \frac{1-t^2}{3}$ then Vieta's System of equations

1) $\begin{cases} x+y+z = 1 \\ xy+yz+zx = p \\ xyz = q \end{cases}$ is solvable in real $x, y, z \geq 0$ iff

$$\max\left\{0, \frac{(1+t)^2(1-2t)}{27}\right\} \le q \le \frac{(1-t)^2(1+2t)}{27} .$$

Remark 1.

Trying to use for q more simple upper bounds, such as $p^2/3$ and $p^2/(4-3p)$ don't led to success.

Remark 2.

Inequality of the problem can be considered as refinement of Weitzenböck's inequality $a^2+b^2+c^2 \ge 4\sqrt{3}F$, which is different from Hadwiger–Finsler inequality

Indeed, $\frac{1}{2}(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-\frac{3}{2} \ge \frac{9}{2}-\frac{3}{2} = 3.$